CS 70 Discrete Mathematics and Probability Theory
Summer 2017 Hongling Lu, Vrettos Moulos, and Allen Tang **HW 4**

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Solve the Rainbow

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

(d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?

(e) If you take three Skittles from the bag, what is the probability that they are all the same color?

2 Probability Potpourri

Prove a brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) If $|\Omega| = n$, how many distinct events does the probability space have?

(c) Find some probability space Ω and three events A, B, and $C \subseteq \Omega$ such that $\mathbb{P}(A) > \mathbb{P}(B)$ and $\mathbb{P}(A \mid C) < \mathbb{P}(B \mid C)$.

(d) If two events C and D are disjoint and $\mathbb{P}(C) > 0$ and $\mathbb{P}(D) > 0$, can C and D be independent? If so, provide an example. If not, why not?

(e) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(f) Two six sided dice are rolled. Find three events such that they are all pairwise independent, but aren't mutually independent.

3 Identity Theft

A group of n friends go to the gym together, and while they are playing basketball, they leave their bags against the nearby wall. An evildoer comes, takes the student ID cards from the bags, randomly rearranges them, and places them back in the bags, one ID card per bag. What is the probability that no one receives his or her own ID card back? [*Hint*: Use the generalized inclusion-exclusion principle.] Then, find an approximation for the probability as $n \to \infty$.

4 Cookie Jars

You have two jars of cookies, jar 1 and jar 2. Each jar starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$). One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X denote the number of remaining cookies in the two jars. What is the distribution of X?

5 Exploring the Geometric Distribution

In this question, we will further investigate the geometric distribution. Let X, Y be i.i.d. geometric random variables with parameter p. Let $U = \min\{X,Y\}$ and $V = \max\{X,Y\} - \min\{X,Y\}$. Compute the joint distribution of (U,V) and prove that U and V are independent. [*Hint*: If $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ are independent, then $\min\{X,Y\} \sim \text{Geometric}(p+q-pq)$.]

6 Poisson Coupling

Consider the following discrete joint distribution for $p \in [0,1]$.

$$\begin{aligned}
&\mathbb{P}(X=0,Y=0) = 1-p, \\
&\mathbb{P}(X=1,Y=y) = \frac{e^{-p}p^y}{y!}, && y=1,2,\ldots, \\
&\mathbb{P}(X=1,Y=0) = e^{-p}-(1-p), \\
&\mathbb{P}(X=x,Y=y) = 0, && \text{otherwise.}
\end{aligned}$$

(a) Recall that all valid distributions satisfy two important properties. Argue that this distribution is a valid joint distribution.

(b) Show that X has the Bernoulli distribution with probability p.

(c) Show that Y has the Poisson distribution with parameter $\lambda = p$.

(d) Show that $\mathbb{P}(X \neq Y) \leq p^2$.

Now, let X_i, $i = 1,2,\ldots$ be a sequence of Bernoulli random variables with probabilities p_i, $i = 1,2,\ldots$. Similarly, let Y_i be a Poisson random variable with parameter $\lambda = p_i$, $i = 1,2,\ldots$. The X_i and Y_i are coupled, so that they have the joint distribution described above (with $p = p_i$), but for $i \neq j$, (X_i,Y_i) and (X_j,Y_j) are independent.

We will now introduce a coupling argument which shows that the distribution of $\sum_{i=1}^n X_i$ approaches a Poisson distribution with parameter $\lambda = p_1 + \cdots + p_n$.

(e) A common way to measure the "distance" between two probability distributions is known as the total variation norm, and it is given by

$$d(X,Y) = \frac{1}{2}\sum_{k=0}^{\infty} |\mathbb{P}(X=k) - \mathbb{P}(Y=k)|.$$

Show that $d(X,Y) \leq \mathbb{P}(X \neq Y)$. [*Hint*: Use the Law of Total Probability to split up the events according to $\{X=Y\}$ and $\{X \neq Y\}$.]

(f) Show that $\mathbb{P}(\sum_{i=1}^n X_i \neq \sum_{i=1}^n Y_i) \leq \sum_{i=1}^n \mathbb{P}(X_i \neq Y_i)$. [*Hint*: Maybe try the Union Bound.]

(g) Finally, for the X_i and Y_i defined above, show that $d(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) \leq \sum_{i=1}^n p_i^2$.

7 Joint Distributions

(a) Suppose that X_i, $i = 1,\ldots,n$ are binary-valued random variables. How many parameters are required to parameterize the joint distribution $\mathbb{P}(X_1 = x_1,\ldots,X_n = x_n)$?

(b) Continuing from the previous part, suppose that all X_is are independent. How many parameters are required to parameterize the joint distribution?

8 Indicator Variables

(a) After throwing n balls into m bins at random, what is the expected number of bins that contains exactly k balls?

(b) Alice and Bob each draw k cards out of a deck of 52 distinct cards with replacement. Find k such that the expected number of common cards that both Alice and Bob draw is at least 1.

(c) How many people do you need in a room so that you expect that there is going to be a shared birthday on a Monday of the year (assume 52 Mondays in a year and 365 days in a year)?