CS 70	Discrete Mathematics and Probability Theory	
Summer 2017	Hongling Lu, Vrettos Moulos, and Allen Tang	HW 3

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many 10-bit strings are there that contain exactly 4 ones?

(b) How many ways are there to arrange n 1s and k 0s into a sequence?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.

How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

(g) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(i) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(j) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

2 Counting on Graphs

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself.

(b) How many distinct cycles are there in a complete graph with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are equivalent up to reflection and rotating vertices.

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

3 Fibonacci Fashion

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \dots, S_t$, where $S_1 \subseteq \{1,2,\dots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\dots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \ge 0} \sum_{x_2 \ge 0} \cdots \sum_{x_t \ge 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

4 Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

(g) Suppose now the coin is flipped $n \ge 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. [*Hint*: The answer is NOT $1/2$.]

5 Calculate These... or Else

(a) A straight is defined as a 5 card hand such that the card values can be arranged in consecutive ascending order, i.e. $\{8, 9, 10, J, Q\}$ is a straight. Values do not loop around, so $\{Q, K, A, 2, 3\}$ is not a straight. However, an ace counts as both a low card and a high card, so both $\{A, 2, 3, 4, 5\}$ and $\{10, J, Q, K, A\}$ are considered straights. When drawing a 5 card hand, what is the probability of drawing a straight from a standard 52-card deck?

(b) What is the probability of drawing a straight or a flush? (A flush is five cards of the same suit.)

(c) When drawing a 5 card hand, what is the probability of drawing at least one card from each suit?

(d) Two squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?

(e) 8 rooks are placed randomly on an 8×8 chessboard. What is the probability none of them are attacking each other? (Two rooks attack each other if they are in the same row, or in the same column).

6 Throwing Balls into a Depth-Limited Bin

Say you want to throw n balls into n bins with depth $k-1$ (they can fit $k-1$ balls, after that the bins overflow). Suppose that n is a large number and $k = 0.1n$. You throw the balls randomly into the bins, but you would like it if they don't overflow. You feel that you might expect not too many balls to land in each bin, but you're not sure, so you decide to investigate the probability of a bin overflowing.

(a) Focus on the first bin. Get an upper bound the number of ways that you can throw the balls into the bins such that this bin overflows. Try giving an argument about the following strategy: select k balls to put in the first bin, and then throw the remaining balls randomly. You should assume that the balls are distinguishable.

(b) Calculate an upper bound on the probability that the first bin will overflow.

(c) Upper bound the probability that some bin will overflow. [*Hint*: Use the union bound.]

(d) How does the above probability scale as n gets really large? [*Hint*: Use the union bound.]