Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Optimal Partners

In the notes, we proved that the SMA always outputs the male-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every man with his best choice results in a pairing at all. Prove by contradiction that no two men can have the same optimal partner. (Note: your proof should not rely on the fact that the SMA outputs the male-optimal pairing.)

2 Relaxed Timing

Suppose that when running the SMA, we relax the rules for the men, so that each unpaired man proposes to the next woman on his list at a time of his choice (some men might procrastinate for several days, while others might propose and get rejected several times in a single day). Can the order of the proposals change the resulting pairing? Give an example of such a change or prove that the pairing that results is the same.

3 Connectivity

Prove the following claims regarding connectivity:

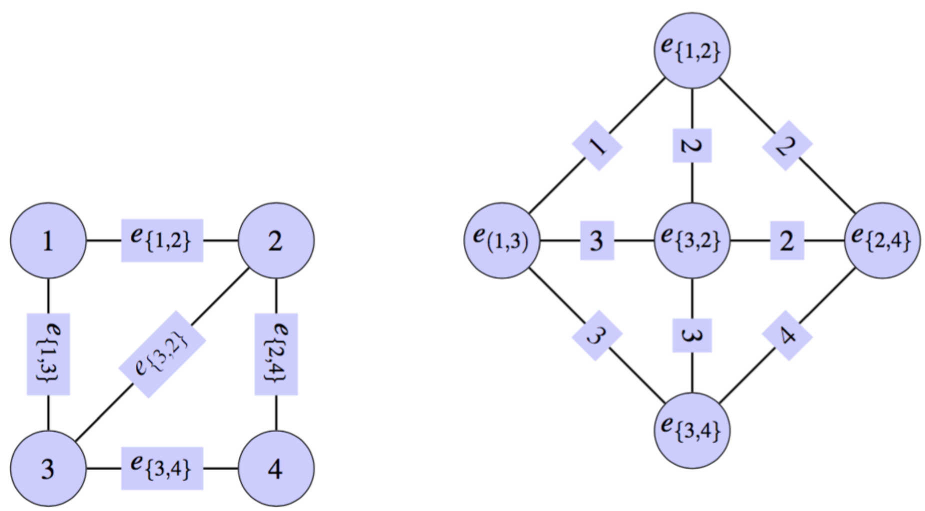
(a) If G is a graph with n vertices such that for any two non-adjacent vertices u, v, it holds that $\deg u + \deg v \geq n-1$, then G is connected.

(b) Give an example to show that if the condition $\deg u + \deg v \geq n-1$ is replaced with $\deg u + \deg v \geq n-2$, then G is not necessarily connected.

(c) For a graph G with n vertices, if the minimum degree of each vertex is at least $n/2$, then G is connected.

(d) If there are exactly two vertices with odd degrees in a graph, then they must be connected to each other (meaning, there is a path connecting these two vertices).

4 Edge Complement



The **edge complement** graph of a graph $G = (V, E)$ is a graph $G' = (V', E')$, such that $V' = E$, and $(i, j) \in E'$ if and only if i and j had a common vertex in G. In the above picture, the graph on the right is the edge complement of the graph on the left: for every edge $e_{\{i,j\}}$ in the graph on the left there is a vertex in the graph on the right. If two edges $e_{\{i,j\}}$ and $e_{\{j,k\}}$ share a vertex j on the left, then the corresponding vertices on the right have an edge j connecting them.

(a) Prove or disprove: if a graph G has an Eulerian tour, then its **edge complement** graph has an Eulerian tour.

(b) Prove or disprove: if a graph's **edge complement** graph G' has an Eulerian tour, then graph G has an Eulerian tour.

5 Always, Sometimes, or Never

In each part below, you are given some information about a graph G. Using only the information in the current part, say whether G will always be planar, always be non-planar, or could be either.

If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) G can be vertex-colored with 4 colors.

(b) G requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of G and v is the number of vertices of G.

(d) G is connected, and each vertex in G has degree at most 2.

(e) Each vertex in G has degree at most 2.

6 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L to a vertex in R (so there does not exist an edge that connects two vertices in L or two vertices in R).

(a) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Prove that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$.

(b) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Let s and t denote the average degree of vertices in L and R respectively. Prove that $s/t = |R|/|L|$.

(c) A double of a graph G consists of two copies of G with edges joining the corresponding "mirror" points. Now suppose that G_1 is a bipartite graph, G_2 is a double of G_1, G_3 is a double of G_2, and so on. Show that $\forall n \geq 1$, G_n is bipartite.

(d) Prove that a graph is bipartite if and only if it can be 2-colored.

7 Countability Practice

(a) Prove or disprove: The set of increasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \geq f(y)$) is countable.

(b) Prove or disprove: The set of decreasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \leq f(y)$) is countable.

(c) Is a set of disks in $\mathbb{R}^2$ such that no two disks overlap necessarily countable or possibly uncountable? [A disk is a region in the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$.]

(d) Is a set of circles in $\mathbb{R}^2$ such that no two circles overlap necessarily countable or possibly uncountable? [*Hint*: A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.]

8 Impossible Programs

Show that none of the following programs can exist.

(a) Consider a program P that takes in any program F, input x and output y and returns true if $F(x)$ outputs y and returns false otherwise.

(b) Consider a program P that takes in any program F and returns true if $F(F)$ halts and returns false if it doesn't halt.

(c) Consider a program P that takes in any programs F and G and returns true if F and G halt on all the same inputs and returns false otherwise.