CS 70 Discrete Mathematics and Probability Theory
Summer 2017 Lu, Moulos, and Tang

HW 1

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 Tautologies and Contradictions

A *tautology* is an expression that evaluates to True for all possible combinations of its variables. A *contradiction* is an expression that evaluates to False for all possible combinations of its variables. State whether the following expressions are tautologies, contradictions, or neither. Justify your answers.

(a) $(x \vee y) \vee (x \vee \neg y)$

(b) $(x \vee y) \wedge (\neg(x \wedge y))$

(c) $x \wedge (x \implies y) \wedge (\neg y)$

(d) $x \implies (x \vee y)$

(e) $(x \implies y) \vee (x \implies \neg y)$

(f) $(x \implies y) \wedge (\neg x \implies y) \wedge (\neg y)$

3 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x \in \mathbb{Z})\ ((2 \mid x \vee 3 \mid x) \implies 6 \mid x)$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

4 Contraposition

Consider the statement "if $a+b < c+d$, then $a < c$ or $b < d$".

(a) Prove this statement with a direct proof.

(b) Prove this statement via contraposition.

(c) Which proof type was easier?

5 Pigeonhole Principle

(a) Prove the following statement: If you put $n+1$ balls into n bins, however you want, then at least one bin must contain at least two balls. This is known as the *pigeonhole principle*.

(b) Use the pigeonhole principle to show the following fact: If there are n students at a homework party ($n \geq 2$), then there are at least two students who are friends with exactly the same number of other students in the homework party. Assume that friendships are always mutual.

6 Airport

Suppose that there are $2n+1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, there is exactly one airplane departing from it, and heading towards the closest airport. Prove by induction that there is an airport which none of the airplanes are heading towards.

7 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

8 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 1.\end{aligned}$$

Therefore, the statement is true. $\square$

(b) **Claim:** For all negative integers n, $-1 - 3 - \cdots + (2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.
Inductive Hypothesis: Assume that $-1 - 3 - \cdots + (2n+1) = -n^2$.
Inductive Step: We need to prove that the statement is also true for $n-1$ if it is true for n, that is, $-1 - 3 - \cdots + (2(n-1)+1) = -(n-1)^2$. Starting from the left hand side,

$$\begin{aligned}-1 - 3 - \cdots + (2(n-1)+1) &= (-1 - 3 - \cdots + (2n+1)) + (2(n-1)+1) \\ &= -n^2 + (2(n-1)+1) \quad \text{(Inductive Hypothesis)} \\ &= -n^2 + 2n - 1 \\ &= -(n-1)^2.\end{aligned}$$

Therefore, the statement is true. $\square$

(c) **Claim:** For all positive integers n, $\sum_{i=0}^{n} 2^{-i} \leq 2$.

Proof. We will prove a stronger statement, that is, $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$, by induction on n.
Base Case: $n = 1 \leq 2 - 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.
Inductive Step: We must show that $\sum_{i=0}^{n+1} 2^{-i} = 2 - 2^{-(n+1)}$. Starting from the left hand side,

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^{-i} &= \sum_{i=0}^{n} 2^{-i} + 2^{-(n+1)} \\
&= (2 - 2^{-n}) + 2^{-(n+1)} \quad \text{(Inductive Hypothesis)} \\
&= 2 - 2^{-(n+1)}.
\end{aligned}$$

Since $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n} \leq 2$, the claim is true. □

(d) **Claim:** For all nonnegative integers n, $2n = 0$.

Proof. We will prove by strong induction on n.
Base Case: $2 \times 0 = 0$. It is true for $n = 0$.
Inductive Hypothesis: Assume that $2k = 0$ for all $0 \leq k \leq n$.
Inductive Step: We must show that $2(n+1) = 0$. Write $n+1 = a+b$ where $0 < a, b \leq n$. From the inductive hypothesis, we know $2a = 0$ and $2b = 0$, therefore,

$$2(n+1) = 2(a+b) = 2a + 2b = 0 + 0 = 0.$$

The statement is true. □

(e) **Claim:** Every positive integer $n \geq 2$ has a unique prime factorization.
In other words, let $2 \leq p_1, p_2, \ldots, p_i \leq n$ be all prime numbers that divide n, there is only one unique way to write n as a product of primes,

$$n = p_1^{d_1} \cdot p_2^{d_2} \cdots p_i^{d_i},$$

where $d_1, d_2, \ldots, d_i \in \mathbb{N}$.

Proof. We will prove by strong induction on n.
Base Case: 2 is a prime itself. It is true for $n = 2$.
Inductive Hypothesis: Assume that the statement is true for all $2 \leq k \leq n$.
Inductive Step: We must prove that the statement is true for $n+1$. If $n+1$ is prime, then it itself is a unique prime factorization. Otherwise, $n+1$ can be written as $x \times y$ where $2 \leq x, y \leq n$. From the inductive hypothesis, both x and y have unique prime factorizations. The product of unique prime factorizations is unique, therefore, $n+1$ has a unique prime factorization. □