1 Check Digits: ISBN

In this problem, we'll look at a real-world applications of check-digits.

International Standard Book Numbers (ISBNs) are 10-digit codes ($d_1 d_2 \dots d_{10}$) which are assigned by the publisher. These 10 digits contain information about the language, the publisher, and the number assigned to the book by the publisher. Additionally, the last digit d_{10} is a "check digit" selected so that $\sum_{i=1}^{10} i \cdot d_i \equiv 0 \pmod{11}$. (*Note that the letter X is used to represent the number 10 in the check digit.*)

(a) Suppose you have very worn copy of the (recommended) textbook for this class. You want to list it for sale online but you can only read the first nine digits: 0-07-288008-? (the dashes are only there for readability). What is the last digit? Please show your work, even if you actually have a copy of the textbook.

(b) Wikipedia says that you can determine the check digit by computing $\sum_{i=1}^{9} i \cdot d_i \pmod{11}$. Show that Wikipedia's description is equivalent to the above description.

(c) Prove that changing any single digit of the ISBN will render the ISBN invalid. That is, the check digit allows you to *detect* a single-digit substitution error.

(d) Can you *switch* any two digits in an ISBN and still have it be a valid ISBN? For example, could $01\underline{2}34\underline{5}678X$ and $01\underline{5}34\underline{2}678X$ both be valid ISBNs?

2 Euclid's Algorithm

(a) Use Euclid's algorithm in the lecture note to compute the greatest common divisor of 527 and 323. List the values of x and y of all recursive calls.

(b) Use the extended Euclid's algorithm in the lecture note to compute the multiplicative inverse of 5 mod 27. List the values of x and y and the returned values of all recursive calls.

(c) Find $x \pmod{27}$ if $5x + 26 \equiv 3 \pmod{27}$. You can use the result computed in (b).

(d) True or false? Assume a, b, and c are integers and $c > 0$. If a has no multiplicative inverse mod c, then $ax \equiv b \pmod{c}$ has no solution. Explain your answer.

3 Paper GCD

Given a sheet of paper such as this one, and no rulers, describe a method to find the GCD of the width and the height of the paper. You can fold or tear the paper however you want, and ultimately you should produce a square piece whose side lengths are equal to the GCD.

4 Baby Fermat

Assume that a does have a multiplicative inverse mod m. Let us prove that its multiplicative inverse can be written as $a^k \pmod{m}$ for some $k \geq 0$.

(a) Consider the sequence $a, a^2, a^3, \ldots \pmod{m}$. Prove that this sequence has repetitions.

(b) Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what can you say about $a^{i-j} \pmod{m}$?

(c) Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is k in terms of i and j?