

1 Markov Chain Basics

A Markov chain is a sequence of random variables X_n , $n = 0, 1, 2, \dots$. Here is one interpretation of a Markov chain: X_n is the state of a particle at time n . At each time step, the particle can jump to another state. Formally, a Markov chain satisfies the Markov property:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i), \quad (1)$$

for all n , and for all sequences of states $i_0, \dots, i_{n-1}, i, j$. In other words, the Markov chain does not have any memory; the transition probability only depends on the current state, and not the history of states that have been visited in the past.

- (a) In lecture, we learned that we can specify Markov chains by providing three ingredients: \mathcal{X} , P , and π_0 . What do these represent, and what properties must they satisfy?

- (b) If we specify \mathcal{X} , P , and π_0 , we are implicitly defining a sequence of random variables X_n , $n = 0, 1, 2, \dots$, that satisfies (1). Explain why this is true.

- (c) Calculate $\mathbb{P}(X_1 = j)$ in terms of π_0 and P . Then, express your answer in matrix notation. What is the formula for $\mathbb{P}(X_n = j)$ in matrix form?

2 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathcal{X} = \{1, 2, 3, 4, 5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshoot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4, 5\}$.

3 Consecutive Flips

Suppose you are flipping a standard coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

(a) Construct a Markov chain that describes the situation with a start state and end state.

(b) Given that you have flipped a (Tails, Heads) so far, how many expected number of flips?

(c) What is the expected number of flips from the start state?

4 Reflecting Random Walk

[This question is optional and may not be covered during discussion.]

Alice starts at vertex 0 and wishes to get to vertex n . When she is at vertex 0 she has a probability of 1 of transitioning to vertex 1. For any other vertex i , there is a probability of $1/2$ of transitioning to $i + 1$ and a probability of $1/2$ of transitioning to $i - 1$.

(a) What is the expected number of steps Alice takes to reach vertex n ? Write down the hitting-time equations, but do not solve them yet.

(b) Solve the hitting-time equations. *[Hint: Let R_i denote the expected number of steps to reach vertex n starting from vertex i . As a suggestion, try writing R_0 in terms of R_1 ; then, use this to express R_1 in terms of R_2 ; and then use this to express R_2 in terms of R_3 , and so on. See if you can notice a pattern.]*