CS 70 Discrete Mathematics and Probability Theory
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1 Tightness of Inequalities

(a) Show by example that Markov's inequality is tight; that is, show that given $k > 0$, there exists a discrete non-negative random variable X such that $\mathbb{P}(X \geq k) = \mathbb{E}[X]/k$.

(b) Show by example that Chebyshev's inequality is tight; that is, show that given $k \geq 1$, there exists a random variable X such that $\mathbb{P}(|X - \mathbb{E}[X]| \geq k\sigma) = 1/k^2$, where $\sigma^2 = \operatorname{var} X$.

(c) Show that there is no non-negative discrete random variable $X \neq 0$, that takes values in some finite set $\{v_1, \ldots, v_N\}$, such that for all $k > 0$, Markov's inequality is tight; that is, $\mathbb{P}(X \geq k) = \mathbb{E}[X]/k$.

2 Working with the Law of Large Numbers

(a) A fair coin is tossed and you win a prize if there are more than 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed and you win a prize if there are more than 40% heads. Which is better: 10 tosses or 100 tosses? Explain.

(c) A coin is tossed and you win a prize if there are between 40% and 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

(d) A coin is tossed and you win a prize if there are exactly 50% heads. Which is better: 10 tosses or 100 tosses? Explain.

3 Inequality Practice

(a) X is a random variable such that $X > -5$ and $\mathbb{E}[X] = -3$. Find an upper bound for the probability of X being greater than or equal to -1.

(b) You roll a die 100 times. Let Y be the sum of the numbers that appear on the die throughout the 100 rolls. Use Chebyshev's inequality to bound the probability of the sum Y being greater than 400 or less than 300.

4 Tellers

Imagine that X is the number of customers that enter a bank at a given hour. To simplify everything, in order to serve n customers you need at least n tellers. One less teller and you won't finish serving all of the customers by the end of the hour. You are the manager of the bank and you need to decide how many tellers there should be in your bank so that you finish serving all of the customers in time. You need to be sure that you finish in time with probability at least 95%.

(a) Assume that from historical data you have found out that $\mathbb{E}[X] = 5$. How many tellers should you have?

(b) Now assume that you have also found out that $\mathrm{var}(X) = 5$. Now how many tellers do you need?