CS 70 Discrete Mathematics and Probability Theory
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1 Bijections

Consider the function

$$f(x) = \begin{cases} x, & \text{if } x \geq 1; \\ 3x-2, & \text{if } \frac{1}{2} \leq x < 1; \\ -x, & \text{if } -1 \leq x < \frac{1}{2}; \\ 2x+3, & \text{if } x < -1. \end{cases}$$

(a) If the domain and range of f are $\mathbb{N}$, is f injective (one-to-one), surjective (onto), bijective?

(b) If the domain and range of f are $\mathbb{Z}$, is f injective (one-to-one), surjective (onto), bijective?

(c) If the domain and range of f are $\mathbb{R}$, is f injective (one-to-one), surjective (onto), bijective?

2 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) $\mathbb{N}$, the set of all natural numbers.

(b) $\mathbb{Z}$, the set of all integers.

(c) $\mathbb{Q}$, the set of all rational numbers.

(d) $\mathbb{R}$, the set of all real numbers.

(e) The integers which divide 8.

(f) The integers which 8 divides.

(g) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(h) Computer programs that halt.

(i) Computer programs that always correctly tell if a program halts or not.

(j) Numbers that are the roots of nonzero polynomials with integer coefficients.

3 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program P prints "Hello World!" before running the kth line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.

4 Countability and the Halting Problem

Prove the Halting Problem using the set of all programs and inputs

a) Show that the set of all programs are countable.

b) Show that the set of all inputs are countable.

c) Assume that you have a program that tells you whether or not it halts. Since the set of all programs and the set of all inputs are countable, we can enumerate them and construct the following table.

	x_1	x_2	x_3	x_4	$\dots$
p_1	H	L	H	L	$\dots$
p_2	L	L	L	H	$\dots$
p_3	H	L	H	L	$\dots$
p_4	L	H	L	L	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now write a program that is not within the set of programs in the table above.

d) Find a contradiction in part a and part b to show that the halting problem can't be solved.