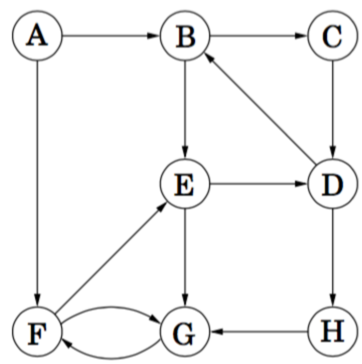
1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $\{(B,C),(C,D),(D,B)\}$

ii. $\{(F,G),(G,F)\}$

iii. $\{(A,B),(B,C),(C,D),(D,B)\}$

iv. $\{(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)\}$

(e) Which of the following are walks in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(A,B),(B,C),(C,D)\}$

v. $\{(E,G),(G,F),(F,G),(G,F)\}$

vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

(f) Which of the following are tours in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

2 Build-Up Error?

What is wrong with the following "proof"?

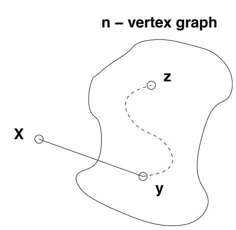
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$.

3 Bipartite Graph

Consider an undirected bipartite graph with two disjoint sets L,R. Prove that a graph is bipartite if and only if it no cycles of odd length.